Euler Theorem Induction proof

Michele Filadelfia

November 16, 2023

1 The problem

Euler's Theorem says that an undirected graph has an Eulerian circuit if and only if every node has even degree. Prove the "only if" part (i.e., "if every node has even degree then an Eulerian circuit exists") by induction.

2 Proof

Proof. **Strong induction on** $|E|$
"If each node has even degree then there exists an Eulerian circuit"

Base case $|E| = 3$:
$G = (V, E)$
$V = \{v1, v2, v3\}$
$E = \{(v1, v2), (v2, v3)(v3, v1)\}$

Inductive step $|E'| = m + 1$:
Inductive hypothesis: "In a graph with $|E| \leq m$ edges, if each node has even degree then there exists an Eulerian circuit"
Thesis: "in a graph with $|E'| = m + 1$ edges, if each node has even degree then there exists an Eulerian circuit"

$G' = (V', E')$
We randomly choose a node $k \in V'$, starting from this node we traverse G' by marking the crossed edges and randomly choosing among the unmarked edges continuing as long as possible.
We can see that at each node we have a trail and that the procedure ends at k, otherwise the graph would not have all the nodes of even degree, so we have a circuit.
Now, we call E" the set of arcs we have traversed.
Then we create a new graph $G'' = (V', E'')$, this new graph has an Eulerian circuit $k, a_1, a_2, ..., a_i, k$, now we have two cases:

Case $E' = E''$:
G' has an Eulerian circuit

Case $E' \neq E''$:
We construct the graph $G^3 = (V', E' \smallsetminus E'')$, observing that $E' = E'' \cup (E' \smallsetminus E'')$, and that $E'' \subseteq E'$ and $(E' \smallsetminus E'') \subseteq E'$.
Given that $|E' \smallsetminus E''| < |E'|$ we can exploit the inductive hypothesis to say that G^3 has an Eulerian circuit $u, b_1, b_2, ..., b_j, u$ with $u \in V'$
Noting that the initial graph is connected, we can finally construct an Eulerian circuit $k, a_1, a_2, ..., u, b_1, b_2, ..., b_j, u, ..., a_i, k$ in G'.

$\square$